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# Pareto optimality and Walrasian equilibria

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**Abstract**

The paper concerns the study of a class of convex, constrained multiobjective optimization problems from the viewpoint of the existence issues. The main feature of the presented approach is that the classical qualification condition requiring the existence of interior points in the effective domains of functions under consideration does not hold. A variant of duality theory for multiobjective optimization problems based on the Fenchel theorem is formulated. Next, by using very recent results on the Walrasian general equilibrium model of economy obtained in Naniewicz [Z. Naniewicz, Pseudo-monotonicity and economic equilibrium problem in reflexive Banach space, Math. Oper. Res. 32 (2007) 436–466] the conditions ensuring the existence of Pareto optimal solutions for the class of multiobjective optimization problems are established. The concept of the proper efficiency is used as the solution notion. Finally, a new version of the second fundamental theorem of welfare economics is presented.


*Keywords:* Pareto optimality; Variational inequality; Walrasian equilibrium; Duality

## 1. Introduction

The aim of this paper is to present some existence results for convex multiobjective optimization problems of the form

$$\left.\begin{array}{l} \text{v-Minimize } f(x) = \big(f_j(x)\big) = \big(f_1(x), \ldots, f_m(x)\big) \\ \text{subject to} \quad x \in \Sigma \subset X. \end{array}\right\} \tag{$\widetilde{P}$}$$

To achieve this goal the existence results obtained for the general equilibrium model of economy in reflexive Banach spaces in Naniewicz [22] are applied. The notation "v-Minimize" refers to a vector minimum problem for which the solution is understood as the so-called properly efficient vector $x \in \Sigma$ which is a minimizer of the scalarized problem

$$\sum_{j=1}^{m} \lambda_j f_j(x) \mapsto \min, \quad x \in \Sigma,$$

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for some $(\lambda_j) \in \mathrm{Int}(\mathbb{R}^m_+)$. The objective functions $f_j$ are assumed to be convex, lower semicontinuous and proper. The results obtained refer to the case in which the objective functions do not satisfy the classical (based on the existence of interior points) qualification condition.

Using some ideas from [29–31] where the duality theory for multiobjective optimization problems has been derived by applying the Fenchel–Rockafellar perturbation technique, we construct the variant of the duality theory based on the Fenchel theorem (see [2,3,24]). As in the aforementioned papers, to any Pareto optimal (properly efficient) vector there corresponds $m$-tuple of dual variables called here an allocation. It provides optimality conditions in terms of the conjugates of the objective functions under consideration. The duality scheme developed here allows to look at a price vector of the general equilibrium model of economy (see [1,15]) as a Pareto optimal (properly efficient) vector for a certain multiobjective optimization problem. This vector is expressed in terms of the conjugates of disutility functions in question. In consequence, it is possible to establish some new existence results for a class of multiobjective optimization problems in which the effective domains of conjugates of the objective functions need not to have nonempty interiors. This case is of importance when dealing with multiobjective optimization problems in reflexive Banach spaces where the positive cones (orthants) have empty interiors—$L^p$ spaces, for instance.

There is an abundant literature on various aspects of duality in multiobjective optimization problems. In this respect we refer the reader to [5,9,10,12,27,28,30,31] and references therein. The general theory of multiobjective optimization can be found in the monographs [6,11,12,14,23] and the bibliographies there.

It is worth to pay special attention to certain new directions in the theory of nonconvex multiobjective optimization based on such mathematical tools like the extremal principle, the generalized differential, the set-valued counterparts of Ekeland variational principle (see [4,16–19] and the references quoted there).

## 2. Fenchel duality and Pareto optima

Let $X$ be a reflexive Banach space with its dual $X^\star$. The pairing over $X^\star \times X$ will be denoted by $\langle\cdot,\cdot\rangle$. Let $\Sigma \subset X$ be a closed, convex, nonempty subset of $X$. Assume

$$f_j : X \to \mathbb{R} \cup \{+\infty\}, \quad j = 1, \ldots, m, \tag{1}$$

to be convex, proper and lower semicontinuous functions.

We study a multiobjective optimization problem of the form:

$$\left.\begin{array}{l} \text{v-Minimize } f(x) = \big(f_j(x)\big) = \big(f_1(x), \ldots, f_m(x)\big) \\ \text{subject to} \quad x \in \Sigma. \end{array}\right\} \tag{$\widetilde{P}$}$$

**Definition 1.** $x^\star \in \Sigma$ is said to be a properly efficient solution (vector) of $(\widetilde{P})$ if there exists $\alpha = (\alpha_j) \in \mathrm{Int}(\mathbb{R}^n_+)$ such that $x^\star$ is a solution of the scalarized problem:

$$\left.\begin{array}{l} \text{Minimize } \displaystyle\sum_{j=1}^m \alpha_j f_j(x) \\ \text{subject to} \quad x \in \Sigma. \end{array}\right\} \tag{$P_\alpha$}$$

In the terminology of [12] such $x^\star$ is called a properly Edgeworth–Pareto solution of $(\widetilde{P})$.

Define the support function of $\Sigma$ by $\Psi(\tau) := \sup_{y\in\Sigma}\langle\tau, y\rangle$, $\tau \in X^\star$. For a finite collection of functions $f_j : X \to \overline{\mathbb{R}}$ define its infimal convolution as

$$\mathop{\square}_{j=1}^{m} f_j(x) = \inf_{\sum_{j=1}^m x_j = x} \sum_{j=1}^m f_j(x_j), \quad x \in X. \tag{2}$$

Recall that $\square_{j=1}^m f_j$ as being convex may fail to be lower semicontinuous, however

$$\Big(\mathop{\square}_{j=1}^{m} f_j\Big)^\star(\tau) = \sum_{j=1}^m f_j^\star(\tau), \quad \tau \in X^\star, \tag{3}$$

where $f_j^\star$ stands for the conjugate of $f_j$. In general, we have

$$\left(\sum_{j=1}^{m} f_j\right)^\star(\tau) \leqslant \left(\mathop{\square}_{j=1}^{m} f_j^\star\right)(\tau), \quad \tau \in X^\star. \tag{4}$$

The infimal convolution $\square_{j=1}^{m} f_j^\star$ is said to be exact at $\tau \in X^\star$ if there is a decomposition $\tau = \sum_{j=1}^{m} \tau_j$ such that

$$\left(\sum_{j=1}^{m} f_j\right)^\star(\tau) = \left(\mathop{\square}_{j=1}^{m} f_j^\star\right)(\tau) = \sum_{j=1}^{m} f_j^\star(\tau_j).$$

One of the well-known consequences of the convex duality theory is the fact that if $\mathrm{Int}(\bigcap_{j=1}^{m} \mathrm{Dom}(f_j)) \neq \emptyset$, then $\square_{j=1}^{m} f_j^\star$ is exact at each point of $X^\star$ (cf. Proposition 3.4, p. 43 [3], see also [25] and [2]).

Our first result reads as follows.

**Theorem 2.** *Suppose that vector $x^\star \in \Sigma$ is a properly efficient solution of $(\widetilde{P})$, i.e. a solution of the scalarized problem $(P_\alpha)$ for some $\alpha \in \mathrm{Int}(\mathbb{R}_+^m)$. Moreover, assume the qualification condition*

$$0 \in \left(\bigcap_{j=1}^{m} \mathrm{Int}\big(\mathrm{Dom}(f_j)\big)\right) - \Sigma. \tag{$CQ_1$}$$

*Then there exists a $m$-tuple $(\pi_j) \in (X^\star)^m$ associated with $x^\star$, such that*

$$\left.\begin{aligned} &\text{(i)} \quad \sum_{j=1}^{m} \pi_j \in \partial\, \mathrm{ind}_\Sigma\big(x^\star\big); \\ &\text{(ii)} \quad -\frac{1}{\alpha_j}\pi_j \in \partial f_j\big(x^\star\big) \quad \textit{for each } j \in \{1, \dots, m\}; \\ &\text{(iii)} \quad (\pi_j) \textit{ minimizes } \sum_{j=1}^{m} \alpha_j f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) + \Psi\left(\sum_{j=1}^{m} \pi_j\right) \longrightarrow \min. \end{aligned}\right\} \tag{5}$$

**Definition 3.** The $m$-tuple $(\pi_j)$ will be called the dual allocation associated with $x^\star$, or dual allocation of $(\widetilde{P})$.

**Proof.** Suppose that vector $x^\star \in \Sigma$ is a properly efficient solution of $(\widetilde{P})$ corresponding to $\alpha \in \mathrm{Int}(R_+^m)$. Hence when setting $\phi(x) := \sum_{j=1}^{m} \alpha_j f_j(x)$, $x \in \Sigma$, we can formulate the problem

$$v := \inf_{x \in X}\big\{\phi(x) + \mathrm{ind}_\Sigma(x)\big\} = \phi\big(x^\star\big) + \mathrm{ind}_\Sigma\big(x^\star\big) = \sum_{j=1}^{m} \alpha_j f_j\big(x^\star\big). \tag{6}$$

According to the Fenchel duality theory (cf. [2,3]) the dual of (6) is formulated as follows:

$$v^\star := \inf_{\tau \in X^\star}\big\{\phi^\star(-\tau) + \Psi(\tau)\big\} = \inf_{\tau \in X^\star}\left\{\left(\sum_{j=1}^{m}(\alpha_j f_j)\right)^\star(-\tau) + \Psi(\tau)\right\}. \tag{7}$$

Note that $(CQ_1)$ implies $\bigcap_{j=1}^{m} \mathrm{Int}(\mathrm{Dom}(\alpha_j f_j)) \neq \emptyset$ which ensures that the infimal convolution $\square_{j=1}^{m}(\alpha_j f_j)^\star$ is exact. Hence for any $\tau \in X^\star$ there exists a decomposition $\tau = \sum_{j=1}^{m} \tau_j$ with

$$\left(\sum_{j=1}^{m}(\alpha_j f_j)\right)^\star(-\tau) = \left(\mathop{\square}_{j=1}^{m}(\alpha_j f_j)^\star\right)(-\tau) = \sum_{j=1}^{m}(\alpha_j f_j)^\star(-\tau_j) = \sum_{j=1}^{m} \alpha_j f_j^\star\left(-\frac{1}{\alpha_j}\tau_j\right).$$

Moreover, $(CQ_1)$ guarantees that $0 \in \operatorname{Int}(\bigcap_{j=1}^m \operatorname{Dom}(f_j) - \Sigma)$, thus by the Fenchel theorem (cf. [3]) there exists $\pi \in X^\star$ with a decomposition $\pi = \sum_{j=1}^m \pi_j$ such that

$$\left.\begin{aligned} &v^\star = \sum_{j=1}^m \alpha_j f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) + \Psi(\pi), \\ &v + v^\star = 0. \end{aligned}\right\} \tag{8}$$

Combining (6) and (8) yields

$$\sum_{j=1}^m \alpha_j f_j\left(x^\star\right) + \sum_{j=1}^m \alpha_j f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) + \Psi(\pi) = 0, \tag{9}$$

which can be written as

$$\sum_{j=1}^m \alpha_j f_j\left(x^\star\right) + \sum_{j=1}^m \alpha_j f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) + \sum_{j=1}^m \langle \pi_j, x^\star\rangle - \langle \pi, x^\star\rangle + \Psi(\pi) = 0. \tag{10}$$

Further, from the Fenchel inequality it follows

$$f_j\left(x^\star\right) + f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) \geqslant \left\langle -\frac{1}{\alpha_j}\pi_j, x^\star\right\rangle, \quad j \in \{1, \dots, m\}, \tag{11}$$

$$\Psi(\pi) \geqslant \langle \pi, x^\star\rangle. \tag{12}$$

Thus in view of (10) we get the equalities

$$f_j\left(x^\star\right) + f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) = \left\langle -\frac{1}{\alpha_j}\pi_j, x^\star\right\rangle, \quad j \in \{1, \dots, m\}, \tag{13}$$

$$\Psi(\pi) = \langle \pi, x^\star\rangle \tag{14}$$

which are equivalent to

$$-\frac{1}{\alpha_j}\pi_j \in \partial f_j\left(x^\star\right), \quad j \in \{1, \dots, m\}, \tag{15}$$

$$\sum_{j=1}^m \pi_j = \pi \in \partial \operatorname{ind}_\Sigma\left(x^\star\right). \tag{16}$$

This implies easily (5). The proof is complete. □

**Corollary 4.** *From Theorem* 2 *it follows that the sufficient condition for the multiobjective optimization problem* $(\widetilde{P})$ *to admit a properly efficient solution is that* $\exists(\alpha_j) \in \operatorname{Int}(\mathbb{R}_+^m)$ *and* $\exists(\pi_j) \in (X^\star)^m$ *such that*

$$0 \in \partial\Psi\left(\sum_{j=1}^m \pi_j\right) - \bigcap_{j=1}^m \partial f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right), \tag{17}$$

*or equivalently,*

$$0 \in \bigcup_{\substack{(\beta_j)\in\operatorname{Int}(\mathbb{R}_+^m)\\ (\tau_j)\in(X^\star)^m}} \left[\partial\Psi\left(\sum_{j=1}^m \tau_j\right) - \bigcap_{j=1}^m \partial f_j^\star\left(-\frac{1}{\beta_j}\tau_j\right)\right]. \tag{18}$$

**Corollary 5.** *The special case occurs when* $\Sigma = X$ (*no constraints, or constraints incorporated in the definition of objective functions*). *Then* $\Psi = \operatorname{ind}_{\{0\}}$ *and because of* $\partial \operatorname{ind}_{\{0\}}(0) = X^\star$, (17) *in Corollary* 4 *reduces to the statement*:

*the sufficient condition for the multiobjective optimization problem* $(\widetilde{P})$ *to admit a properly efficient solution is that there exist an allocation* $(\pi_j) \in (X^\star)^m$ *with* $\sum_{j=1}^m \pi_j = 0$ *and* $(\alpha_j) \in \mathrm{Int}(\mathbb{R}^m_+)$ *such that*

$$\bigcap_{j=1}^m \partial f_j^\star\left(-\frac{1}{\alpha_j}\pi_j\right) \neq \emptyset. \tag{19}$$

**Remark 6.** The condition (19) can be regarded as Fermat's role for the multiobjective optimization problem under consideration.

**Remark 7.** An allocation $(\pi_j)$ in Theorem 2 constitutes a minimizer for the function defined in (5)(iii). This suggests introducing the parametrized family of scalar minimization problems $(P^\star_\alpha)_{\alpha\in\mathrm{Int}(\mathbb{R}^m_+)}$ by

$$\left.\begin{array}{l}\text{Minimize } f^\star_\alpha(\tau) := \left(\displaystyle\sum_{j=1}^m(\alpha_j f_j)\right)^\star(-\tau) + \Psi(\tau)\\ \text{subject to} \quad \tau = (\tau_j) \in \big(X^\star\big)^m.\end{array}\right\} \tag{$P^\star_\alpha$}$$

To each scalar optimization problem $(P^\star_\alpha)$ one can assign the following multiobjective optimization problem

$$\left.\begin{array}{l}\text{v-Minimize } \tilde{f}^\star_\alpha(\tau) := \left(\Psi\left(\displaystyle\sum_{j=1}^m \tau_j\right), f_1^\star\left(-\frac{1}{\alpha_1}\tau_1\right), \ldots, f_m^\star\left(-\frac{1}{\alpha_m}\tau_m\right)\right)\\ \text{subject to} \quad \tau = (\tau_j) \in \big(X^\star\big)^m.\end{array}\right\} \tag{$\widetilde{P}^\star_\alpha$}$$

**Definition 8.** $\pi = (\pi_j) \in (X^\star)^m$ will be called the properly efficient allocation of $(\widetilde{P}^\star_\alpha)$ if it is a solution of the following scalarized problem:

$$\left.\begin{array}{l}\text{Minimize } f^\star_{\alpha,\beta}(\tau) := \displaystyle\sum_{j=1}^m \beta_j f_j^\star\left(-\frac{1}{\alpha_j}\tau_j\right) + \Psi\left(\sum_{j=1}^m \tau_j\right)\\ \text{subject to} \quad \tau = (\tau_j) \in \big(X^\star\big)^m,\end{array}\right\} \tag{$P^\star_{\alpha,\beta}$}$$

for some $\beta \in \mathrm{Int}(\mathbb{R}^m_+)$.

From the Fenchel theorem we get immediately the following results concerning the existence of properly efficient solutions to $(\widetilde{P})$ and $(\widetilde{P}^\star_\alpha)$.

**Theorem 9.** *If for some* $\alpha \in \mathrm{Int}(\mathbb{R}^m_+)$ *the qualification condition*

$$0 \in \mathrm{Int}\left(\sum_{j=1}^m \mathrm{Dom}\big(\alpha_j f_j^\star\big) + \mathrm{Dom}\,\Psi\right), \tag{$CQ_2$}$$

*then* $(\widetilde{P})$ *has at least one properly efficient vector.*

**Proof.** In the framework of Fenchel's theory the dual of the minimization problem $(P^\star_\alpha)$ takes the form

$$\sum_{j=1}^m \alpha_j f_j(x) + \mathrm{ind}_\Sigma(x) \longrightarrow \min, \tag{20}$$

and according to the Fenchel theorem the hypothesis $(CQ_2)$ ensures its solvability (see [3]). This completes the proof. □

**Theorem 10.** *If* $(CQ_1)$ *holds, then for each* $\alpha \in \mathrm{Int}(\mathbb{R}^m_+)$ *the problem* $(\widetilde{P}^\star_\alpha)$ *has at least one properly efficient allocation.*

**Proof.** Note that for each $\alpha \in \mathrm{Int}(\mathbb{R}^m_+)$, $(CQ_1)$ implies

$$0 \in \mathrm{Int}\left(\mathrm{Dom}\left(\sum_{j=1}^{m} \alpha_j f_j\right) - \Sigma\right)$$

which by the Fenchel theorem ensures that the scalar minimization problem

$$\left(\sum_{j=1}^{m} (\alpha_j f_j)\right)^{\star}(\tau) + \Psi(\tau) \mapsto \min$$

has a solution $\pi$. From $(CQ_1)$ it follows also that the infimal convolution $\square_{j=1}^{m}(\alpha_j f_j)^{\star}$ is exact. Hence the decomposition results $\pi = \sum_{j=1}^{m} \pi_j$ from which by the procedure already explained we conclude that $(\pi_j)$ is a solution of $(P_\alpha^{\star})$. Consequently, the allocation $(\pi_j)$ is a solution of $(\widetilde{P}_\alpha^{\star})$, as desired. □

**Remark 11.** Note that the qualification condition $(CQ_2)$ in Theorem 9 is too strong in many important cases because it requires the existence of interior points in effective domains of conjugates of the objective functions under consideration.

In the next section an important class of problems will be shown, where $(CQ_2)$ can be replaced by weaker hypotheses, still ensuring the existence of solutions for the corresponding multiobjective optimization problems.

## 3. Competitive equilibria and Pareto optimality

Assume $\mathcal{K} \subset X$ to be a closed, convex, pointed cone (i.e. $\mathcal{K} \cap (-\mathcal{K}) = 0$) with the associated positive dual $\mathcal{K}^+ = \{\tau \in X^{\star}\colon\ \langle \tau, x\rangle \geqslant 0\ \forall x \in \mathcal{K}\}$. It must be stressed that $\mathcal{K}$ is not required to have nonempty interior. Let $V_j : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$, $j = 1, \dots, m$, be convex lower semicontinuous functions. From now on the notation will be used: $\overline{V}_j := V_j + \mathrm{ind}_{\mathcal{K}}$. Further, let $\phi_j : \mathcal{K}^+ \to \mathbb{R}_+$ be convex, continuous, positively homogeneous of degree 1 functions with nonnegative values on $\mathcal{K}^+$. Set $\Phi = \sum_{j=1}^{m} \phi$. Since $\Phi : \mathcal{K}^+ \to \overline{\mathbb{R}}$ is a convex, lower semicontinuous, positively homogeneous of degree 1 function with $\mathrm{Dom}(\Phi) = \mathcal{K}^+$, there exists a nonempty convex closed subset $\Delta \subset X$ such that $\Delta \supset \Delta - \mathcal{K}$ and $\Phi(\tau) = \sup_{y \in \Delta}\langle \tau, y\rangle$, $\tau \in \mathcal{K}^+$.

In [22] the following problem related to the general equilibrium model of economy has been studied (see also [21]):
Find $\pi \in \mathcal{K}^+$ and $x_j \in \mathcal{K}$, $j = 1, \dots, m$, such as to satisfy the conditions:

$$V_j(x_j) = \min\bigl\{V_j(x)\colon\ \langle \pi, x\rangle \leqslant \phi_j(\pi),\ x \in \mathcal{K}\bigr\}, \quad j = 1, \dots, m, \tag{PM}$$

$$\left\langle \tau - \pi, -\sum_{j=1}^{m} x_j \right\rangle + \Phi(\tau) - \Phi(\pi) \geqslant 0, \quad \forall \tau \in \mathcal{K}^+, \tag{PE}$$

and the following existence result has been established:

**Theorem 12.** *(See [22].) Let $X$ be a reflexive Banach space. Suppose that there exists $\mu > 0$ such that*

$$\|x + y\| \geqslant \mu\bigl(\|x\| + \|y\|\bigr), \quad \forall x, y \in \mathcal{K}$$

*and $\Delta \cap \mathcal{K} \subset B_X(0, C) = \{y \in X\colon\ \|y\| \leqslant C\}$ for some $C > 0$. Let $F_0$ be a finite dimensional subspace of $X$. Assume that for any $j \in \{1, \dots, m\}$ the following hypotheses hold*:

(A$_1$) $0 \in \mathrm{cl}(\mathrm{Dom}(\overline{V}_j))$;

(A$_2^1$) *One of the two conditions below holds*:
- (i) *For any $\tau \in \mathcal{K}^+ \setminus \{0\}$ with $\phi_j(\tau) = 0$, if $-t^n\tau^n \in \partial \overline{V}_j(x^n)$, $t^n \to +\infty$ in $\mathbb{R}$, $\tau^n \rightharpoonup \tau$ weakly in $X^{\star}$, $x^n \rightharpoonup x$ weakly in $X$ and $\langle \tau^n, x^n\rangle \to 0$, then $\liminf_{n\to\infty} \overline{V}_j^{\star}(-t^n\tau^n) > -\infty$,*
- (ii) *or $\mathrm{Dom}(\overline{V}_j)$ is closed.*

*Moreover, assume that*

(A$_3$) $\bigcup_{j=1}^{m} \partial \overline{V}_j^\star(0) \subset F_0$;
(A$_4$) $\Phi(\tau) = \sum_{j=1}^{m} \phi_j(\tau) \geqslant \gamma\|\tau\|$, $\forall \tau \in \mathcal{K}^+$, $\gamma > 0$;
(A$_5$) *For at least one* $j \in \{1, \dots, m\}$, $0 \notin \partial \overline{V}_j(0)$;
(A$_6$) $(\sum_{j=1}^{m} \partial \overline{V}_j^\star(0)) \cap \Delta = \emptyset$.

*Then there exists* $(\pi, (x_j), (\alpha_j), r) \in \mathcal{K}^+ \times \mathcal{K}^m \times (\mathbb{R}_+ \cup \{+\infty\})^m \times (0, 1]$ *with* $\pi \neq 0$, *such that*

$$\left.\begin{array}{l} \left.\begin{array}{l} -\alpha_j \pi \in \partial \overline{V}_j(x_j) \\ \langle \pi, x_j \rangle - \phi_j(\pi) \in \partial \operatorname{ind}_{\geqslant 0}(\alpha_j) \end{array}\right\} \quad \text{if } \alpha_j \in \mathbb{R}_+, \\ \left.\begin{array}{l} -\pi \in \partial^\infty \overline{V}_j(x_j) \\ \langle \pi, x_j \rangle = \phi_j(\pi) = 0 \end{array}\right\} \quad \text{if } \alpha_j = +\infty, \\ j \in \Xi \quad \Rightarrow \quad \alpha_j > 0, \\ \Phi(\tau) - \Phi(\pi) \geqslant \left\langle \tau - \pi, \dfrac{1}{r} \displaystyle\sum_{j=1}^{m} x_j \right\rangle \quad \forall \tau \in \mathcal{K}^+, \end{array}\right\} \tag{$PW$}$$

*where*

$$\Xi = \left\{ j \in \{1, \dots, m\}\colon\ \langle \tau, y \rangle > \phi_j(\tau) \text{ for any } \tau \in \mathcal{K}^+ \setminus \{0\} \text{ and } y \in \partial \overline{V}_j^\star(0) \text{ if } \partial \overline{V}_j^\star(0) \neq \emptyset,\ \text{or } \partial \overline{V}_j^\star(0) = \emptyset \right\}. \tag{21}$$

Here $\partial^\infty \varphi : X \to 2^{X^\star}$ stands for the asymptotic subdifferential of $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, defined for $x \in \operatorname{Dom}(\varphi)$ by

$$\partial^\infty \varphi(x) = \left\{ \tau \in X^\star\colon\ (\tau, 0) \in N_{\operatorname{epi}\varphi}\big(x, \varphi(x)\big) \right\},$$

where $N_{\operatorname{epi}\varphi}(x, \varphi(x))$ stands for the normal cone to the epigraph of $\varphi$ at $(x, \varphi(x)) \in \operatorname{epi}\varphi$ [8,25,26]. We also refer the reader to the variational approach to economic equilibrium problems developed in [13].

Now we impose hypotheses ensuring all the Lagrange multipliers to be positive and finite, i.e. $\alpha_j \in (0, +\infty)$ for each $j = 1, \dots, m$. It can be achieved if (cf. [22])

(A$_7$) $\Xi = \{1, \dots, m\}$

and (A$_2^1$) is replaced by

(A$_2$) For any $-\beta^n \tau^n \in \partial \overline{V}_j(x^n)$ such that $\beta^n \to +\infty$, $\|\tau^n\| = 1$ and $\{x^n\}$ bounded, the implication below holds:

$$\text{If } \langle \tau^n, x^n \rangle \to 0 \quad \text{then } \liminf_{n \to \infty} \phi_j\big(\tau^n\big) > 0. \tag{22}$$

Under the hypotheses (A$_2$) and (A$_7$) all the $\alpha_j$'s in ($PW$) are positive and finite, consequently $r = 1$.

Accordingly, from Theorem 12 we are led to the existence result that will be used in our further investigations.

**Theorem 13.** *Let X be a reflexive Banach space. Suppose that there exists* $\mu > 0$ *such that*

$$\|x + y\| \geqslant \mu\big(\|x\| + \|y\|\big), \quad \forall x, y \in \mathcal{K}$$

*and* $\Delta \cap \mathcal{K} \subset B_X(0, C)$ *for some* $C > 0$. *Let* $F_0$ *be a finite dimensional subspace of X. Suppose that the hypotheses* (A$_1$)–(A$_7$) *hold. Then there exists* $(\pi, (x_j), (\alpha_j)) \in \mathcal{K}^+ \times \mathcal{K}^m \times \operatorname{Int}(\mathbb{R}_+^m)$ *with* $\pi \neq 0$, *such that*

$$\left.\begin{array}{l} -\alpha_j \pi \in \partial \overline{V}_j(x_j), \\ \langle \pi, x_j \rangle - \phi_j(\pi) = 0, \\ \Phi(\tau) - \Phi(\pi) \geqslant \left\langle \tau - \pi, \displaystyle\sum_{j=1}^{m} x_j \right\rangle \quad \forall \tau \in \mathcal{K}^+. \end{array}\right\} \tag{$PW_0$}$$

**Definition 14.** Following the economic terminology, the vector $\pi$ and allocation $(x_j)$ satisfying $(PW_0)$ will be called the competitive (Walrasian) equilibrium [1,15].

From $(PW_0)$ it follows

$$\frac{1}{\alpha_j}V_j(x_j)+\frac{1}{\alpha_j}\overline{V}_j^\star(-\alpha_j\pi)=-\langle\pi,x_j\rangle,\quad j=1,\dots,m,$$
$$\Phi(\pi)=\sum_{j=1}^m\phi_j(\pi)=\sum_{j=1}^m\langle\pi,x_j\rangle=\left\langle\pi,\sum_{j=1}^m x_j\right\rangle, \tag{23}$$

from which we have

$$\sum_{j=1}^m\frac{1}{\alpha_j}V_j(x_j)+\mathrm{ind}_\Delta\left(\sum_{j=1}^m x_j\right)+\sum_{j=1}^m\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star(-\pi)+\Phi(\pi)=0. \tag{24}$$

It is easy to notice (thanks to $(\frac{1}{\alpha_j}\overline{V}_j)^\star(\cdot)=\frac{1}{\alpha_j}\overline{V}_j^\star(\alpha_j\cdot)$) that this equality is related to the minimization problems:

$$X\ni y\longmapsto\left(\sum_{j=1}^m\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star\right)^\star(y)+\mathrm{ind}_\Delta(y)\longrightarrow\min \tag{25}$$

and its dual

$$X^\star\ni\tau\longmapsto\sum_{j=1}^m\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star(-\tau)+\Phi(\tau)\longrightarrow\min, \tag{26}$$

and means that $y=\sum_{j=1}^m x_j$ and $\tau=\pi$ are their solutions, respectively (cf. [3]).

**Corollary 15.** *If $(\pi,(x_j))\in X^\star\times X^m$ is a competitive equilibrium, then the price vector $\pi$ is:*

1. *Minimizer of the scalar optimization problem:*

$$\left.\begin{aligned}&\textit{Minimize } F_\alpha^\star(\pi):=\sum_{j=1}^m\frac{1}{\alpha_j}\overline{V}_j^\star(-\alpha_j\pi)+\Phi(\pi)\\&\textit{subject to}\quad \pi\in\mathcal{K}^+\setminus\{0\}.\end{aligned}\right\} \tag{$Q_\alpha^\star$}$$

2. *Properly efficient solution of the multiobjective optimization problem:*

$$\left.\begin{aligned}&\textit{v-Minimize } \widetilde{F}_\alpha^\star(\pi):=\big(\Phi(\pi),\overline{V}_1^\star(-\alpha_1\pi),\dots,\overline{V}_m^\star(-\alpha_m\pi)\big)\\&\textit{subject to}\quad \pi\in\mathcal{K}^+\setminus\{0\}.\end{aligned}\right\} \tag{$\widetilde{Q}_\alpha^\star$}$$

*Moreover, the allocation $x=(x_j)$ is:*

1. *A solution of the scalar minimization problem:*

$$\left.\begin{aligned}&\textit{Minimize } F_\alpha(x):=\sum_{j=1}^m\frac{1}{\alpha_j}\overline{V}_j(x_j)+\mathrm{ind}_\Delta\left(\sum_{j=1}^m x_j\right)\\&\textit{subject to}\quad (x_j)\in\mathcal{K}^m.\end{aligned}\right\} \tag{$Q_\alpha$}$$

2. *A properly efficient solution of the multiobjective optimization problem:*

$$\left.\begin{aligned}&\textit{v-Minimize } \widetilde{F}(x)=\left(\mathrm{ind}_\Delta\left(\sum_{j=1}^m x_j\right),\overline{V}_1(x_1),\dots,\overline{V}_m(x_m)\right)\\&\textit{subject to}\quad (x_j)\in\mathcal{K}^m.\end{aligned}\right\} \tag{$\widetilde{Q}$}$$

**Remark 16.** It must be emphasized that in comparison with the currently used methods the hypotheses of Theorem 13 do not require:

(i) The preference relations determined by dis-utility functions $\overline{V}_j$ to be locally nonsatiated (minimizers of $\overline{V}_j$ are allowed).
(ii) The existence of interior points in the effective domains of functions under consideration and in the cone $\mathcal{K}$.
(iii) Any variant of the condition called $\omega$-properness [1,7,15] imposed on preferences to compensate for the absence of interior points in the positive cone $\mathcal{K}$.

On the basis of Theorem 13 we are allowed to formulate some existence results for a class of multiobjective optimization problems.

**Corollary 17.** *Assume all the hypotheses of Theorem* 13. *Then there exists* $\alpha \in \operatorname{Int}(\mathbb{R}^m_+)$ *such that* $(\widetilde{Q}^\star_\alpha)$ *has at least one nontrivial properly efficient vector.*

**Corollary 18.** *Assume all the hypotheses of Theorem* 13. *Then there exists* $\alpha \in \operatorname{Int}(\mathbb{R}^m_+)$ *such that* $(\widetilde{Q}_\alpha)$ *has at least one properly efficient allocation.*

## 4. Existence of Pareto optimal solutions

Now consider the case of $\overline{V}^\star_j$, $j = 1, \ldots, m$, being positive homogeneous of degree $\sigma_j > 0$, $j = 1, \ldots, m$, respectively. Since $\pi$ is a solution of $(Q^\star_\alpha)$, we get

$$\sum_{j=1}^{m} \alpha_j^{\sigma_j - 1} \overline{V}^\star_j(-\pi) + \Phi(\pi) \longrightarrow \min. \tag{27}$$

Accordingly, $\pi$ is a properly efficient optimal solution of the multiobjective optimization problem:

$$\left.\begin{aligned} &\text{v-Minimize } \big(\Phi(\pi), \overline{V}^\star_1(-\pi), \ldots, \overline{V}^\star_m(-\pi)\big) \\ &\text{subject to} \quad \pi \in \mathcal{K}^+ \setminus \{0\}. \end{aligned}\right\} \tag{$\widetilde{Q}^\star$}$$

Thus we are led to the existence result for multiobjective optimization problem involving objective functions positive homogeneous of an arbitrary positive degree.

**Theorem 19.** *Assume all the hypotheses of Theorem* 13. *Moreover, suppose that* $\overline{V}^\star_j$ *are positive homogeneous of degree* $\sigma_j > 0$, $j = 1, \ldots, m$. *Then the multiobjective optimization problem* $(\widetilde{Q}^\star)$ *admits at least one non trivial properly efficient vector* $\pi \in \mathcal{K}^+$ $(\pi \neq 0)$. *Moreover, there exist* $\alpha \in \operatorname{Int}(\mathbb{R}^m_+)$ *and an allocation* $(x_j)_{j=1}^m \in \mathcal{K}^m$ *such that*

$$\left.\begin{aligned} &\text{(i)} \quad \pi \in \partial \operatorname{ind}_\Delta\Bigg(\sum_{j=1}^m x_j\Bigg); \\ &\text{(ii)} \quad -\alpha_j \pi \in \partial \overline{V}_j(x_j) \quad \textit{for each } j = 1, \ldots, m. \end{aligned}\right\} \tag{28}$$

*Consequently,* $x = (x_j)$ *is a properly efficient allocation for the multiobjective problem* $(\widetilde{Q})$ *and a solution for the scalarized minimization problem* $(Q_\alpha)$.

**Remark 20.** If $\operatorname{Int}(\mathcal{K}) = \emptyset$, then the counterpart of the qualification condition $(CQ_2)$ for the multiobjective optimization problem $(\widetilde{Q}^\star)$ does not hold and Theorem 9 does not apply. This happens when $X = L^p(\Omega)$, $1 < p < \infty$, and $\mathcal{K} = \{x \in X\colon x \geqslant 0, \text{ a.e. in } \Omega\}$, for instance.

Theorem 13 can also be seen as the existence result for the multiobjective optimization problem $(\widetilde{Q})$.

**Theorem 21.** *Assume all the hypotheses of Theorem* 13. *Then the problem* $(\widetilde{Q})$ *admits at least one properly efficient allocation* $x = (x_j) \in \mathcal{K}^m$.

Now consider the multiobjective optimization problem $(\widetilde{Q}^\star)$ without the assumption that $\overline{V}_j^\star$ are positive homogeneous. At the beginning of the study suppose that $\overline{V}_j$ do not have minimizers, i.e. $\partial\overline{V}_j^\star(0)=\emptyset$ for each $j=1,\dots,m$. Moreover, $\phi_j(\tau)=\frac{1}{m}\Phi(\tau)$ $\forall\tau\in\mathcal{K}^+$. As previously, assume that $\Phi(\tau)\geqslant\gamma\|\tau\|$, $\tau\in\mathcal{K}^+$. Hence $\phi_j(\tau)\geqslant\gamma_j\|\tau\|$ with $\gamma_j=\frac{\gamma}{m}$. Under the foregoing hypotheses Theorem 13 takes the form

**Theorem 22.** *Let $X$ be a reflexive Banach space. Suppose that there exists $\mu>0$ such that*

$$\|x+y\|\geqslant\mu\big(\|x\|+\|y\|\big),\quad \forall x,y\in\mathcal{K},\tag{29}$$

*and $\Delta\cap\mathcal{K}\subset B_X(0,C)=\{y\in X:\ \|y\|\leqslant C\}$ for some $C>0$. Assume that for any $j\in\{1,\dots,m\}$ the following hypotheses hold*:

($H_1$) $0\in\mathrm{cl}(\mathrm{Dom}(\overline{V}_j))$;

($H_2$) $\Phi(\tau)\geqslant\gamma\|\tau\|$ $\forall\tau\in\mathcal{K}^+$, $\gamma>0$;

($H_3$) $\partial\overline{V}_j^\star(0)=\emptyset$.

*Then there exists $(\pi,(x_j),(\alpha_j))\in\mathcal{K}^+\times\mathcal{K}^m\times\mathrm{Int}(\mathbb{R}^m_+)$ with $\pi\neq0$, such that $(PW_0)$ holds with $\phi_j=\frac{1}{m}\Phi$, $j=1,\dots,m$.*

**Remark 23.** Without loss of generality one can suppose that $\pi$ in Theorem 22 fulfills the requirement that $\|\pi\|=c>0$ with arbitrarily chosen positive $c$. The constant $c$ will be specified later when necessary.

Now, for $s=(s_j)\in\mathrm{Int}(\mathbb{R}^m_+)$ define $\overline{V}_j^s(y):=\overline{V}_j(s_jy)$, $y\in\mathcal{K}$, $j=1,\dots,m$. It is easy to see that the new defined functions fulfill all the requirements of Theorem 22. Therefore there exists $(\pi^s,(x_j^s),(\alpha_j^s))\in\mathcal{K}^+\times\mathcal{K}^m\times\mathrm{Int}(\mathbb{R}^m_+)$ with $\|\pi^s\|=c$, $c>0$, such that

$$\left.\begin{aligned}
&-\frac{\alpha_j^s}{s_j}\pi^s\in\partial\overline{V}_j\big(s_jx_j^s\big),\\
&\big\langle\pi^s,x_j^s\big\rangle-\phi_j\big(\pi^s\big)=0,\\
&\Phi(\tau)-\Phi\big(\pi^s\big)\geqslant\left\langle\tau-\pi^s,\sum_{j=1}^m x_j^s\right\rangle\quad\forall\tau\in\mathcal{K}^+.
\end{aligned}\right\}\tag{$PW_s$}$$

It leads to the statement that

$$\left.\begin{aligned}
&\sum_{j=1}^m\frac{1}{\alpha_j^s}\overline{V}_j\big(s_jx_j^s\big)+\mathrm{ind}_\Delta\left(\sum_{j=1}^m x_j^s\right)\mapsto\min,\\
&\sum_{j=1}^m\frac{1}{\alpha_j^s}\overline{V}_j^\star\left(-\frac{\alpha_j^s}{s_j}\pi^s\right)+\Phi\big(\pi^s\big)\mapsto\min.
\end{aligned}\right\}\tag{30}$$

Thus in order to show that $(\widetilde{Q}^\star)$ has a solution it is sufficient to arrange conditions under which the Lagrange multipliers $(\alpha_j^s)$ coincide with $(s_j)$. In other words, we have to ensure that a mapping (possibly multivalued) which assigns to $(s_j)$ the corresponding Lagrange multipliers $(\alpha_j^s)$ possesses a fixed point.

Let us recall that $\Phi^\star=\mathrm{ind}_\Delta$ with $\Delta\cap\mathcal{K}\subset B_X(0,C)$. From (29) it follows that if $\sum_{j=1}^m x_j\in\Delta$, $(x_j)\in\mathcal{K}^m$, then $\|x_j\|<\frac{C}{m\mu^{m-1}}$ for each $j=1,\dots,m$.

Now we claim that under the additional assumption

($H_4$) $\partial\overline{V}_j(0)=\emptyset$, $j=1,\dots,m$, there exist $\sigma_j \in (0,\infty)$, $j = 1,\dots,m$, with the property that

$$s = (s_j) \in \prod_{j=1}^{m} [\sigma_j, \infty) \quad \Longrightarrow \quad \left(\alpha_j^s\right) \in \prod_{j=1}^{m} [\sigma_j, \infty). \tag{31}$$

Here $\prod_{j=1}^m [\sigma_j,\infty)$ stands for the Cartesian $m$-product of $[\sigma_j,\infty)$. On the contrary, suppose that this is not true. Thus, for some $j \in \{1,\dots,m\}$ there exists a sequence $s^n \in \prod_{j=1}^m (0,\infty)$ for which $\alpha_j^{s^n} < s_j^n$ with $s_j^n \to 0$ as $n \to \infty$. Taking into account that $\frac{\alpha_j^{s^n}}{s_j^n} < 1$, $-\frac{\alpha_j^{s^n}}{s_j^n}\pi^{s^n} \in \partial \overline{V}_j(s_j^n x_j^{s^n})$, $\|\pi^{s^n}\| = c$ and $\|x_j\| \leqslant \frac{C}{m\mu^{m-1}}$ one can suppose that $s_j^n x_j^{s^n} \to 0$ in $X$, and $\frac{\alpha_j^{s^n}}{s_j^n}\pi^{s^n} \rightharpoonup \tilde{\pi}$ weakly in $X^\star$ for some $\tilde{\pi} \in \mathcal{K}^+$. By the maximal monotonicity this leads to $-\tilde{\pi} \in \partial \overline{V}_j(0)$ what contradicts the assumption $(\mathrm{H}_4)$. The claim has been established.

Further, since $\overline{V}_j$ are convex proper and lower semicontinuous, there exist $a_j, b_j \geqslant 0$ with

$$\overline{V}_j(y) \geqslant -a_j - b_j\|y\|, \quad \forall y \in \mathcal{K},\ j = 1,\dots,m. \tag{32}$$

From $(PW_s)_1$ and $(\mathrm{H}_1)$ it follows

$$\overline{V}_j(y) - \overline{V}_j(s_j x_j^s) \geqslant \left\langle -\frac{\alpha_j^s}{s_j}\pi^s, y - s_j x_j^s \right\rangle, \quad \forall y \in \mathcal{K},$$

which thanks to $\sum_{j=1}^m x_j^s \in \Delta$ and $\langle \pi^s, x_j^s\rangle = \phi_j(\pi^s)$ implies

$$\begin{aligned}\frac{c\gamma}{m}\alpha_j^s = c\gamma_j\alpha_j^s \leqslant \alpha_j^s\phi_j\left(\pi^s\right) &= \alpha_j^s\left\langle \pi^s, x_j^s\right\rangle \leqslant \overline{V}_j(s_j y) - \overline{V}_j\left(s_j x_j^s\right) + \alpha_j^s\left\langle \pi^s, y\right\rangle \\ &\leqslant \overline{V}_j(s_j y) + \alpha_j^s c\|y\| + a_j + b_j s_j \frac{C}{m\mu^{m-1}}, \quad \forall y \in \mathrm{Dom}(\overline{V}_j).\end{aligned}$$

Fix $y_j \in \mathrm{Dom}(\overline{V}_j)$ in such a way to fulfill the requirements $\|y_j\| \leqslant \frac{\gamma}{2m}$ and $\overline{V}_j(s_j y_j) \leqslant d_j + 1$ where

$$d_j := \max\Big\{0, \inf_{\substack{\|y\| \leqslant \frac{\gamma}{2m} \\ s \in [\sigma_j, +\infty)}} \overline{V}_j(sy)\Big\}.$$

Note that due to $(\mathrm{H}_1)$, $d_j < \infty$. Consequently, we are led to the estimate

$$\alpha_j^s \leqslant \frac{2m(d_j+1) + 2ma_j + 2b_j s_j \frac{C}{\mu^{m-1}}}{c\gamma}. \tag{33}$$

Now we are ready to precise the value of the constant $c$ as mentioned in Remark 23. Choose $c > 0$ with $2b_j\frac{C}{\mu^{m-1}} < c\gamma$ for each $j = 1,\dots,m$. This allows to find $S_j > 0$ such that

$$\frac{2m(d_j+1) + 2ma_j + 2b_j S_j \frac{C}{\mu^{m-1}}}{c\gamma} \leqslant S_j.$$

Indeed it is enough to take

$$S_j > \frac{2m(d_j+1) + 2ma_j}{c\gamma - 2b_j\frac{C}{\mu^{m-1}}}. \tag{34}$$

From now on we thus assume that

$$c > \max_{j=1,\dots,m}\{b_j\}\frac{2C}{\gamma\mu^{m-1}}. \tag{35}$$

Under the foregoing choice of $c$, whenever $s_j \in [\sigma_j, S_j]$ the corresponding Lagrange multipliers $\alpha_j^s \in [\sigma_j, S_j]$, $j = 1,\dots,m$.

Next let us recall that to each $s = (s_j) \in \prod_{j=1}^m [\sigma_j, S_j]$ there correspond $(\pi^s, (x_j^s), (\alpha_j^s)) \in \mathcal{K}^+ \times \mathcal{K}^m \times \mathrm{Int}(\mathbb{R}_+^m)$, $\|\pi^s\| = c$, such that $(PW_s)$ holds. Our assumption here is that $\pi^s$ is unique, i.e.

(H$_5$) The mapping $\prod_{j=1}^m[\sigma_j,S_j]\ni s\mapsto\{\pi^s\colon\ \|\pi^s\|=c$ and $(\pi^s,(x_j^s),(\alpha_j^s))$ fulfills $(PW_s)\}$ is single-valued.

Since the set of all Lagrange multipliers corresponding to $\pi^s$ is closed, convex, bounded and nonempty, we conclude that the mapping $\Lambda$ has closed, convex, bounded and nonempty values (for the detailed study of this assertion we refer to [20–22]).

Finally, our study will be devoted to establishing the upper semicontinuity of the mapping $\Lambda:\prod_{j=1}^m[\sigma_j,S_j]\to\prod_{j=1}^m[\sigma_j,S_j]$ which assigns to any $s=(s_j)\in\prod_{j=1}^m[\sigma_j,S_j]$ the set of all corresponding Lagrange multipliers $\Lambda(s)=\{\alpha^s=(\alpha_j^s)\}\subset\prod_{j=1}^m[\sigma_j,S_j]$ fulfilling $(PW_s)$ with $\|\pi^s\|=c$.

For this purpose suppose that $s^n\to s$ as $n\to\infty$, $(\alpha_j^{s^n})\in\Lambda(s^n)$ and $\alpha_j^{s^n}\to\alpha_j^s$. It will be proved that $\alpha_j^s\in\Lambda(s)$. By the assumption we have

$$\left.\begin{aligned}&-\frac{\alpha_j^{s^n}}{s_j^n}\pi^{s^n}\in\partial\overline{V}_j\big(s_j^nx_j^{s^n}\big),\\&\big\langle\pi^{s^n},x_j^{s^n}\big\rangle-\phi_j\big(\pi^{s^n}\big)=0,\\&\Phi(\tau)-\Phi\big(\pi^{s^n}\big)\geqslant\left\langle\tau-\pi^{s^n},\sum_{j=1}^m x_j^{s^n}\right\rangle\quad\forall\tau\in\mathcal{K}^+,\end{aligned}\right\}\tag{$PW_{s^n}$}$$

for some $x_j^{s^n}$ and $\pi^{s^n}$ with $\|\pi^{s^n}\|=c$. By the boundedness argument we can assume that

$$\begin{aligned}&\pi^{s^n}\rightharpoonup\pi^s,\quad\text{weakly in }X^\star,\\&x_j^{s^n}\rightharpoonup x_j^s,\quad\text{weakly in }X,\end{aligned}$$

for some $\pi^s\in\mathcal{K}^+$ and $x_j^s\in\mathcal{K}$. From $(PW_{s^n})_1$ it follows

$$\overline{V}_j\big(s_j^ny\big)-\overline{V}_j\big(s_j^nx_j^{s^n}\big)\geqslant\left\langle-\frac{\alpha_j^{s^n}}{s_j^n}\pi^{s^n},y-x_j^{s^n}\right\rangle,\quad\forall y\in\mathcal{K}.$$

Hence, by substituting $y=\frac{s_j}{s_j^n}x_j^s$, passing to the limit and taking into account the weak lower semicontinuity of $\overline{V}_j$ we obtain

$$0\geqslant\overline{V}_j\big(s_jx_j^s\big)-\liminf_{n\to\infty}\overline{V}_j\big(s_j^nx_j^{s^n}\big)\geqslant\limsup_{n\to\infty}\left\langle-\frac{\alpha_j^{s^n}}{s_j^n}\pi^{s^n},\frac{s_j}{s_j^n}x_j^s-x_j^{s^n}\right\rangle=\frac{\alpha_j^s}{s_j}\limsup_{n\to\infty}\big\langle-\pi^{s^n},x_j^s-x_j^{s^n}\big\rangle,$$

which leads to

$$\big\langle\pi^s,x_j^s\big\rangle\geqslant\limsup_{n\to\infty}\big\langle\pi^{s^n},x_j^{s^n}\big\rangle,\quad j=1,\dots,m.\tag{36}$$

From $(PW_{s^n})_3$ with $\tau=\pi^s$ substituted we easily get

$$\left\langle\pi^s,\sum_{j=1}^m x_j^s\right\rangle\leqslant\liminf_{n\to\infty}\left\langle\pi^{s^n},\sum_{j=1}^m x_j^{s^n}\right\rangle.$$

Combining this with (36) we obtain

$$\big\langle\pi^s,x_j^s\big\rangle=\lim_{n\to\infty}\big\langle\pi^{s^n},x_j^{s^n}\big\rangle,\quad j=1,\dots,m.\tag{37}$$

Therefore, thanks to the weak lower semicontinuity of $\Phi$ combined with $(PW_{s^n})_3$ we conclude that $\lim_{n\to\infty}\Phi(\pi^{s^n})=\Phi(\pi^s)$. Since $\phi_j=\frac{1}{m}\Phi$ we get $\lim_{n\to\infty}\phi_j(\pi^{s^n})=\phi_j(\pi^s)$. The maximal monotonicity of $\partial\overline{V}_j(\cdot)$ together with (37) allows the conclusion $(PW_s)_1$ (cf. [3]). Finally, if we assume that

(H$_6$) For any $\{\pi^n\}\subset\mathcal{K}^+$, if $\pi^n\rightharpoonup\pi$ weakly in $X^\star$, $\|\pi^n\|=c$ and $\Phi(\pi^n)\to\Phi(\pi)$, then $\|\pi\|=c$.

Then the upper semicontinuity of $\Lambda$ is easy to be established.

Now we are ready to invoke the Kakutani theorem for the statement that $\Lambda$ has a fixed point, i.e. $s = (s_j) \in \Lambda(s)$ for some $s \in \prod_{j=1}^{m}[\sigma_j, S_j]$. This means that there exist $\pi^s \in \mathcal{K}^+$ with $\|\pi^s\| = c$ and $(x_j^s) \in \mathcal{K}^m$ such that $(PW_s)$ holds with $\alpha_j^s = s_j$. Accordingly, $(PW_s)$ takes the form

$$\left.\begin{aligned} &-\pi^s \in \partial \overline{V}_j\big(s_j x_j^s\big),\\ &\big\langle \pi^s, x_j^s\big\rangle - \phi_j\big(\pi^s\big) = 0,\\ &\Phi(\tau) - \Phi\big(\pi^s\big) \geqslant \left\langle \tau - \pi^s, \sum_{j=1}^{m} x_j^s \right\rangle \quad \forall \tau \in \mathcal{K}^+. \end{aligned}\right\} \tag{38}$$

The obtained result allows the following conclusions: $\pi^s$ and $(x_j^s)$ are solutions of the optimization problems (compare with (30)):

$$\sum_{j=1}^{m} \frac{1}{s_j} \overline{V}_j^\star\big(-\pi^s\big) + \Phi\big(\pi^s\big) \mapsto \min$$

and

$$\sum_{j=1}^{m} \frac{1}{s_j} \overline{V}_j\big(s_j x_j^s\big) + \operatorname{ind}_\Delta\left(\sum_{j=1}^{m} x_j^s\right) \mapsto \min,$$

respectively. By this way we have shown that $(\widetilde{Q}^\star)$ admits at least one solution. The result can be summarized as follows.

**Theorem 24.** *Let $X$ be a reflexive Banach space. Suppose that there exists $\mu > 0$ such that*

$$\|x + y\| \geqslant \mu\big(\|x\| + \|y\|\big), \quad \forall x, y \in \mathcal{K} \tag{39}$$

*and $\Delta \cap \mathcal{K} \subset B_X(0, C) = \{y \in X\colon\ \|y\| \leqslant C\}$ for some $C > 0$. Assume that for any $j \in \{1, \dots, m\}$ the following hypotheses hold*:

(H$_1$) $0 \in \operatorname{cl}(\operatorname{Dom}(\overline{V}_j))$;
(H$_3$) $\partial \overline{V}_j^\star(0) = \emptyset$;
(H$_4$) $\partial \overline{V}_j(0) = \emptyset$.

*Moreover, assume that*

(H$_2$) $\Phi(\tau) \geqslant \gamma\|\tau\|$, $\forall \tau \in \mathcal{K}^+$, $\gamma > 0$;
(H$_5$) *The mapping* $\prod_{j=1}^{m}[\sigma_j, S_j] \ni s \mapsto \{\pi^s\colon\ \|\pi^s\| = c$ *and* $(\pi^s, (x_j^s), (\alpha_j^s))$ *fulfills* $(PW_s)\}$ *is single-valued*;
(H$_6$) *For any* $\{\pi^n\} \subset \mathcal{K}^+$, *if* $\pi^n \rightharpoonup \pi$ *weakly in* $X^\star$, $\|\pi^n\| = c$ *and* $\Phi(\pi^n) \to \Phi(\pi)$, *then* $\|\pi\| = c$.

*Then the multiobjective optimization problem*

$$\left.\begin{aligned} &\text{v-Minimize } \big(\Phi(\pi), \overline{V}_1^\star(-\pi), \dots, \overline{V}_m^\star(-\pi)\big)\\ &\text{subject to} \quad \pi \in \mathcal{K}^+ \setminus \{0\} \end{aligned}\right\} \tag{$\widetilde{Q}^\star$}$$

*has at least one nontrivial properly efficient solution $\pi \in \mathcal{K}^+$ such that $\|\pi\| = c$ with $c$ fulfilling the estimate* (35), *i.e.*

$$\sum_{j=1}^{m} \frac{1}{s_j} \overline{V}_j^\star(-\pi) + \Phi(\pi) \mapsto \min,$$

*for some* $s \in \prod_{j=1}^{m}[\sigma_j, S_j]$. *Moreover, there exists an allocation* $(x_j)_{j=1}^{m} \in \mathcal{K}^m$ *such that*

$$\left.\begin{aligned} &\text{(i)}\quad \pi \in \partial\,\mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right);\\ &\text{(ii)}\quad -\pi \in \partial \overline{V}_j(s_j x_j)\quad \textit{for each } j = 1, \dots, m. \end{aligned}\right\} \tag{40}$$

**Remark 25.** Note that (40) implies

$$\text{(iii)}\quad \sum_{j=1}^{m} \frac{1}{s_j}\overline{V}_j(s_j x_j) + \mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right) \mapsto \min \tag{41}$$

(compare this with (5)(iii)).

**Remark 26.** The results obtained can be applied to a class of multiobjective optimization problems of the form

$$\left.\begin{aligned} &\text{v-Minimize } F(\pi) = \big(F_0(\pi), F_1(\pi), \dots, F_m(\pi)\big)\\ &\text{subject to}\quad \pi \in \mathcal{K}^+ \setminus \{0\}, \end{aligned}\right\} \tag{$P_F$}$$

where $F_j : X^\star \to \overline{\mathbb{R}}$, $j = 0, 1, \dots, m$, is a collection of proper, convex, lower semicontinuous functions, with $F_0$ being additionally positive homogeneous of degree 1, and $\mathcal{K}^+ \subset X^\star$ is a closed convex cone. Theorems 13 and 19 apply when the dual positive cone $\mathcal{K}$ of $\mathcal{K}^+$ fulfills (39). In such circumstances, if we set $F_{j-}(\cdot) := F_j(-\cdot)$, then in order to establish the existence result our task is to check whether $F_0$ and $F^\star_{j-}$ fulfill all the hypotheses imposed on $\Phi$ and $\overline{V}_j$, respectively. Note that the existence of interior points in $\mathcal{K}$ and $\mathcal{K}^+$ is not required.

**Remark 27.** The condition (17) in Corollary 4 related to $(\widetilde{Q}^\star)$ takes the form: there exist an allocation $(x_0, (x_j)) \in (-\Delta) \times \mathcal{K}^m$ and $(1, (s_j)) \in \mathrm{Int}(\mathbb{R}^{m+1}_+)$ such that

$$0 \in \partial\,\mathrm{ind}_{-\mathcal{K}}\left(\sum_{j=1}^{m} x_j + x_0\right) - \left[\bigcap_{j=1}^{m} \partial \overline{V}_{j-}(-s_j x_j) \cap \partial\,\mathrm{ind}_\Delta(-x_0)\right], \tag{42}$$

where $\overline{V}_{j-}(\cdot) := \overline{V}_j(-\cdot)$, $j = 1, \dots, m$.

## 5. Remark on the second fundamental theorem of welfare economics

The second fundamental theorem of welfare economics provides conditions under which a properly efficient allocation $(x_j) \in \mathcal{K}^m$ can be supported as a price equilibrium with transfers [1,15].

Our task now is to formulate conditions on dis-utility functions $\overline{V}_j : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ and budget functions $\phi_j : \mathcal{K}^+ \to \mathbb{R}$ that allow to show the implication: if $(x_j)$ is properly efficient allocation of $\widetilde{Q}$ (a solution of $(Q_\alpha)$ for some $\alpha \in \mathrm{Int}(\mathbb{R}^m_+)$), then there exists $\pi \in \mathcal{K}^+$ such that $(\pi, (x_j))$ constitutes a competitive equilibrium.

**Theorem 28.** *Let $X$ be a reflexive Banach space. Suppose that there exists* $\mu > 0$ *such that*

$$\|x + y\| \geqslant \mu\big(\|x\| + \|y\|\big), \quad \forall x, y \in \mathcal{K} \tag{43}$$

*and* $\Delta \cap \mathcal{K} \subset B_X(0, C)$ *for some* $C > 0$. *Assume* $(x_j)$ *to be a properly efficient allocation of* $\widetilde{Q}$ *(a solution of* $(Q_\alpha)$ *for some* $\alpha \in \mathrm{Int}(\mathbb{R}^m_+)$*) and*

(H1) $\Phi(\tau) \geqslant \gamma\|\tau\|$, $\tau \in \mathcal{K}^+$, $\gamma > 0$.

(H2) *There exist* $a_j, b_j \geqslant 0$ *with* $\sum_{j=1}^{m} b_j < \gamma$ *such that* $\overline{V}^\star_j(-\tau) \geqslant -a_j - b_j\|\tau\|$, $j = 1, \dots, m$.

*Then there exist $\pi \in \mathcal{K}^+$, $\pi \neq 0$, such that $(\pi, (x_j))$ is a competitive equilibrium with transfers in the sense that one can find wealth levels $w_j \geqslant 0$, $j = 1, \dots, m$, for which the following holds*:

$$\left.\begin{aligned} &V_j(x_j) = \min\{V_j(x)\colon \langle \pi, x\rangle \leqslant w_j,\ x \in \mathcal{K}\}, \quad j = 1, \dots, m,\\ &\sum_{j=1}^m x_j \in \partial\Phi(\pi),\\ &\Phi(\pi) = \sum_{j=1}^m w_j. \end{aligned}\right\} \tag{44}$$

**Proof.** In view of (43) and $\Delta \cap \mathcal{K} \subset B_X(0, C)$ for some $C > 0$, one can find $R > 0$ large enough to fulfill the requirements

$$\text{For each } (y_j) \in \mathcal{K}^m\colon \quad \sum_{j=1}^m y_j \in \Delta \quad \Longrightarrow \quad \|y_j\| < R, \quad j = 1, \dots, m,$$

and

$$R > \max_{j=1,\dots,m}\{b_j\} \frac{\gamma}{\sum_{k=1}^m b_k}.$$

Set $\overline{V}_{jR}(\cdot) := \overline{V}_j(\cdot) + \mathrm{ind}_{B_X(0,R)}(\cdot)$. Then $\mathrm{Dom}(\overline{V}^\star_{jR}) = X^\star$ for each $j = 1, \dots, m$. Since the properly efficient allocation $(x_j)$ has the property that $\sum_{j=1}^m x_j \in \Delta$, we have $\|x_j\| < R$. Therefore, by the Fenchel theorem there is a zero duality gap for the two problems in duality:

$$v := \inf_{(y_j)\in\mathcal{K}^m}\left\{\sum_{j=1}^m \frac{1}{\alpha_j}\overline{V}_{jR}(y_j) + \mathrm{ind}_\Delta\left(\sum_{j=1}^m y_j\right)\right\} = \sum_{j=1}^m \frac{1}{\alpha_j}\overline{V}_j(x_j) + \mathrm{ind}_\Delta\left(\sum_{j=1}^m x_j\right) \tag{45}$$

and

$$v^\star := \inf_{\tau\in\mathcal{K}^+}\left\{\sum_{j=1}^m \left(\frac{1}{\alpha_j}\overline{V}_{jR}\right)^\star(-\tau) + \Phi(\tau)\right\}, \tag{46}$$

what means that $v + v^\star = 0$. Taking into account that $0 \in \mathrm{Int}(\mathrm{Dom}(\overline{V}_j) - B_X(0, R))$ we get

$$\begin{aligned} \left(\frac{1}{\alpha_j}\overline{V}_{jR}\right)^\star(-\tau) &= \left(\frac{1}{\alpha_j}\overline{V}_j + \mathrm{ind}_{B_X(0,R)}\right)^\star(-\tau)\\ &= \left(\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star \Box\, \mathrm{ind}^\star_{B_X(0,R)}\right)(-\tau)\\ &= \left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star(-\tau_{j1}) + R\|\tau_{j2}\|\\ &= \frac{1}{\alpha_j}\overline{V}^\star_j(-\alpha_j\tau_{j1}) + R\|\tau_{j2}\|, \quad \tau_{j1} + \tau_{j2} = \tau \in \mathcal{K}^+. \end{aligned} \tag{47}$$

In order to establish the assertion it will be shown that (46) has a solution. From (H1) and (H2) we get the estimates:

$$\begin{aligned} &\sum_{j=1}^m \left(\frac{1}{\alpha_j}\overline{V}_{jR}\right)^\star(-\tau) + \Phi(\tau)\\ &\quad\geqslant \sum_{j=1}^m \frac{1}{\alpha_j}\overline{V}^\star_j(-\alpha_j\tau_{j1}) + R\sum_{j=1}^m \|\tau_{j2}\| + \gamma\|\tau\|\\ &\quad\geqslant -\sum_{j=1}^m \frac{a_j}{\alpha_j} - \sum_{j=1}^m b_j\|\tau_{j1}\| + R\sum_{j=1}^m \|\tau_{j2}\| + \gamma\sum_{j=1}^m \frac{b_j}{\sum_{j=1}^m b_k}\|\tau_{j1} + \tau_{j2}\| \end{aligned}$$

$$\geqslant -\sum_{j=1}^{m}\frac{a_j}{\alpha_j} - \sum_{j=1}^{m} b_j\|\tau_{j1}\| + R\sum_{j=1}^{m}\|\tau_{j2}\| + \gamma\sum_{j=1}^{m}\frac{b_j}{\sum_{k=1}^{m} b_k}\|\tau_{j1}\| - \gamma\sum_{j=1}^{m}\frac{b_j}{\sum_{j=1}^{m} b_k}\|\tau_{j2}\|$$

$$\geqslant -\sum_{j=1}^{m}\frac{a_j}{\alpha_j} + \frac{\gamma - \sum_{k=1}^{m} b_j}{\sum_{k=1}^{m} b_k}\sum_{j=1}^{m} b_j\|\tau_{j1}\| + \sum_{j=1}^{m}\left(R - \frac{\gamma b_j}{\sum_{k=1}^{m} b_k}\right)\|\tau_{j2}\|$$

from which the coercivity of the function $\sum_{j=1}^{m}(\frac{1}{\alpha_j}\overline{V}_{jR})^\star(-\cdot) + \Phi(\cdot)$ follows. Due to the reflexivity of $X^\star$ it has a minimizer, say $\pi \in \mathcal{K}^+$. Hence, by duality,

$$\sum_{j=1}^{m}\frac{1}{\alpha_j}\overline{V}_j(x_j) + \mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right) + \sum_{j=1}^{m}\frac{1}{\alpha_j}\overline{V}^\star_{jR}(-\alpha_j\pi) + \Phi(\pi) = 0 \tag{48}$$

which leads to

$$\sum_{j=1}^{m}\frac{1}{\alpha_j}\overline{V}_j(x_j) + \mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right) + \sum_{j=1}^{m}\frac{1}{\alpha_j}\overline{V}^\star_j(-\alpha_j\pi_{j1}) + R\sum_{j=1}^{m}\|\pi_{j2}\| + \Phi(\pi) = 0. \tag{49}$$

This can be rewritten as

$$\sum_{j=1}^{m}\left(\frac{1}{\alpha_j}\overline{V}_j(x_j) + \frac{1}{\alpha_j}\overline{V}^\star_j(-\alpha_j\pi_{j1}) + \langle\pi_{j1}, x_j\rangle\right) + \Phi(\pi) + \mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right) - \left\langle\pi, \sum_{j=1}^{m} x_j\right\rangle$$
$$+ \sum_{j=1}^{m}\big(R\|\pi_{j2}\| + \langle\pi_{j2}, x_j\rangle\big) = 0.$$

By the Fenchel inequality, for each $j = 1, \dots, m$,

$$\frac{1}{\alpha_j}\overline{V}_j(x_j) + \frac{1}{\alpha_j}\overline{V}^\star_j(-\alpha_j\pi_{j1}) + \langle\pi_{j1}, x_j\rangle \geqslant 0,$$
$$\Phi(\pi) + \mathrm{ind}_\Delta\left(\sum_{j=1}^{m} x_j\right) - \left\langle\pi, \sum_{j=1}^{m} x_j\right\rangle \geqslant 0$$

giving rise to the inequality

$$\sum_{j=1}^{m}\big(R\|\pi_{j2}\| + \langle\pi_{j2}, x_j\rangle\big) \leqslant 0.$$

Since $\|x_j\| < R$, $j = 1, \dots, m$, we easily conclude that $\pi_{j2} = 0$ for each $j = 1, \dots, m$. This implies

$$v^\star := \inf_{\tau\in\mathcal{K}^+}\left\{\sum_{j=1}^{m}\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star(-\tau) + \Phi(\tau)\right\} = \sum_{j=1}^{m}\left(\frac{1}{\alpha_j}\overline{V}_j\right)^\star(-\pi) + \Phi(\pi). \tag{50}$$

Thus we are led to the conclusion that

$$-\alpha_j\pi \in \partial\overline{V}_j(x_j), \quad j = 1, \dots, m,$$
$$\sum_{j=1}^{m} x_j \in \partial\Phi(\pi).$$

Finally, we assert that (44) is fulfilled with the wealth levels $w_j := \langle\pi, x_j\rangle$ for each $j = 1, \dots, m$. This completes the proof. □

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